The golden ratio prediction for the solar neutrino mixing

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Abstract

We present a simple texture that predicts the cotangent of the solar neutrino mixing angle to be equal to the golden ratio. This prediction is 1.4σ below the present best-fit value and final SNO and KamLAND data could discriminate it from tri-bi-maximal mixing. The neutrino mass matrix is invariant under a $Z_2 \otimes Z_2'$ symmetry: that geometrically is a reflection along the diagonal of the golden rectangle. Assuming an analogous structure in the quark sector suggests a golden prediction for the Cabibbo angle, $\theta_C = \pi/4 - \theta_{12} \approx 13.3^\circ$, up to the uncertainties comparable to V_{ub}.

The neutrino mass matrix exhibits large mixing angles and a mild mass hierarchy. If flavor has an underlying simplicity, it might be easier to recognize it from neutrinos, rather than from charged leptons and quarks, where it is difficult to disentangle $\mathcal{O}(1)$ factors from whatever generates their large mass hierarchies. For the moment the atmospheric mixing angle θ_{23} is consistent with maximal and the θ_{13} mixing angle is consistent with zero [1]. The solar neutrino mixing angle is measured to be $\tan^2\theta_{12} = 0.445 \pm 0.045$ [1]: large but not maximal.

Some non-trivial information might therefore be contained in θ_{12}, and several speculations have been invoked to explain its value. Tri-bi-maximal neutrino mixing [2] is the most popular proposal. It is based on the idea that there are both bi-maximal $(0, 1, 1)/\sqrt{2}$ as well as tri-maximal $(1, 1, 1)/\sqrt{3}$ mixings in the lepton sector. This proposal was initially not based on any deep argument, except that it predicts a solar mixing angle $\tan^2\theta_{12} = 0.5$ close to the observed one, and more precisely 1.2σ above the experimental best fit value. Models that try to explain tri-bi-maximal mixing in terms of broken flavour symmetries have been realized later [3].

We propose the following texture for the neutrino Majorana mass matrix m_ν and for the charged lepton Yukawa couplings λ_E:

$$m_\nu = \begin{pmatrix} 0 & m & 0 \\ m & m & 0 \\ 0 & 0 & m_{\rm atm} \end{pmatrix}, \qquad \lambda_E = \begin{pmatrix} \lambda_e & 0 & 0 \\ 0 & \lambda_\mu/\sqrt{2} & \lambda_\tau/\sqrt{2} \\ 0 & -\lambda_\mu/\sqrt{2} & \lambda_\tau/\sqrt{2} \end{pmatrix}. \tag{1}$$

It just assumes some texture zeroes and some strict equalities among different entries. The solar neutrino mixing comes from the neutrino sector, and the atmospheric mixing from the charged lepton sector. The mass eigenstates of the neutrino mass matrix are

$$m_\nu \cdot \begin{pmatrix} 1 \\ \varphi \\ 0 \end{pmatrix} = m\varphi \begin{pmatrix} 1 \\ \varphi \\ 0 \end{pmatrix}, \qquad m_\nu \cdot \begin{pmatrix} -\varphi \\ 1 \\ 0 \end{pmatrix} = -\frac{m}{\varphi} \begin{pmatrix} -\varphi \\ 1 \\ 0 \end{pmatrix}, \qquad m_\nu \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = m_{\rm atm} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\varphi = (1+\sqrt{5})/2 = 1 + 1/\varphi \approx 1.62$ is known as the golden ratio [4]. Thanks to its peculiar mathematical properties this constant appears in various natural phenomena, possibly including solar neutrinos. Indeed the (1,2) entries of our neutrino mass matrix are proportional to the generator of the Fibonacci recursion, $(x,y) \to (y, x+y)$, that describes various growth phenomena with common limit $y/x = \varphi$. The three neutrino mixing angles obtained from eq. (1) are $\theta_{\rm atm} = \pi/4$, $\theta_{13} = 0$ and, more importantly (see also [5]),

$$\tan^2\theta_{12} = 1/\varphi^2 = 0.382, \qquad \text{i.e.} \qquad \sin^2 2\theta_{12} = 4/5, \tag{2}$$

in terms of the parameter $\sin^2 2\theta_{12}$ directly measured by vacuum oscillation experiments, such as KamLAND. This prediction for θ_{12} is 1.4σ below the experimental best fit value. The solar mass splitting, measured to be $\Delta m^2_{\rm sun} = (8.0 \pm 0.3)\ 10^{-5}\,{\rm eV}^2$ [1], in our case is given by $\Delta m^2_{\rm sun} = +\sqrt{5}m^2$, so the lightest neutrino mass is $|m_1| = m/\varphi = (3.7 \pm 0.1)\,{\rm meV}$, and $m_2 = (9.7 \pm 0.4)\,{\rm meV}$. The parameter $|m_{ee}|$, probed by $0\nu2\beta$ decay experiments [1], vanishes. A positive measurement of θ_{13} might imply that our prediction for θ_{12} suffers an uncertainty up to θ_{13}.

Within the see-saw [6] context, the texture in eq. (1) can be e.g. realized from the following texture for right-handed neutrino masses M and neutrino Yukawa couplings λ_ν:

$$M = \begin{pmatrix} -M & M & 0 \\ M & 0 & 0 \\ 0 & 0 & M_{\rm atm} \end{pmatrix}, \qquad \lambda_\nu = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_{\rm atm} \end{pmatrix}, \tag{3}$$

together with λ_E as in eq. (1). The $-$ sign depends on our convention for the phases of N_1, N_2.

The simple form of the textures in eq. (1) and eq. (3) suggests that the successful prediction for θ_{12} may be not just a numerical coincidence, and motivates to realize it within flavor models, that might give extra indications about the uncertain aspects discussed above. The neutrino mass matrix m_ν is invariant under the $L \to R\cdot L$ and $L \to R'\cdot L$ symmetries, where $L = (L_1, L_2, L_3)$ are the three left-handed leptons and

$$R = \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4}$$

are two Z_2 reflections, i.e. they satisfy $\det R = -1$, $R\cdot R^T = 1$ and $R \cdot R = 1$. The first Z_2 is a reflection along the diagonal of the golden rectangle in the $(1,2)$ plane, see Fig. 1. The second Z_2' is the $L_3 \to -L_3$ symmetry. Any neutrino mass matrix is invariant under a $Z_2 \otimes \mathrm{Z}_2'$ symmetry; in our case this symmetry, written in the texture basis, has the relatively simple form in eq. (4).

One can justify eq. (3) by demanding that right-handed neutrinos $N = (N_1, N_2, N_3)$ transform in the same way as left-handed doublets, $N \to RN$ and $N \to R'N$. More precisely, these symmetries only force m_ν and λ_ν, as well as the kinetic matrices for L and N, all to have the same form as M in eq. (3), up to an additional term proportional to the unit matrix. This form of M and λ_E is more general than the simplest form ineq. (1), but is enough to imply the prediction for θ_{12}, provided that λ_E has the form in eq. (1). Therefore the occurrence of $0\nu2\beta$ decay is *allowed* by the symmetries (4) and the neutrino mass eigenvalues are not restricted to satisfy $|m_2/m_1| = \varphi^2$. This allows us to extend the analogous symmetries to the quark sector with very large mass hierarchies.

The $\mathrm{Z}_2 \otimes \mathrm{Z}_2'$ cannot be extended to right-handed leptons $E = (E_1, E_2, E_3)$ in such a way that λ_E has the form in eq. (1). Indeed, there is a generic no-go argument that tells that the prediction

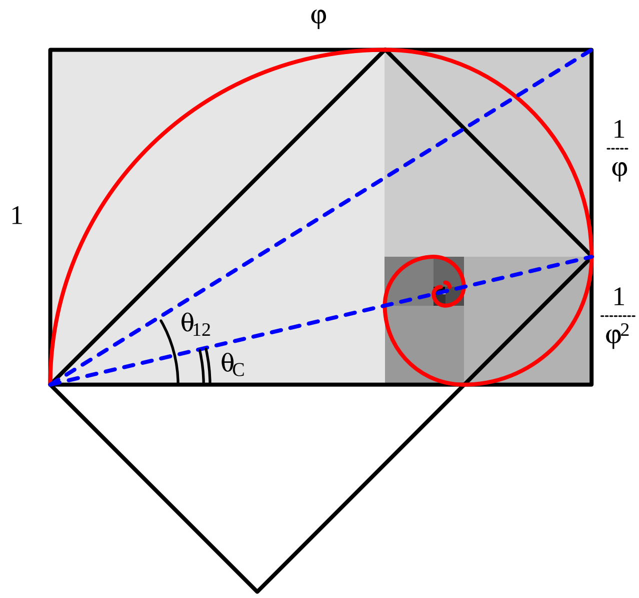

Figure 1: *Geometrical illustration of the connection between our predictions for θ_{12} and θ_C and the golden rectangle. The two dashed lines are the reflection axis of the Z_2 symmetry for the neutrino mass matrix and for the up-quark mass matrix.*

$\theta_{23} = \pi/4$ cannot be implied by any unbroken symmetry [7]: one needs to consider suitably broken symmetries, and this opens a plethora of possibilities. In our case, the lepton Yukawa matrix λ_E has a simple form, diagonal up to a rotation by $\pi/4$ in the (2,3) sector. Therefore such a simple structure of λ_E could be explained in terms of an appropriate pattern of symmetry breaking. Indeed one can extend the $Z_2 \otimes Z_2'$ to E such that in the unbroken limit one has $\lambda_E = 0$ (for example, by assuming $E \to -E$ under both Z_2), and try to obtain the desired small Yukawa couplings of charged leptons by suitably choosing symmetry-breaking effects, e.g. by demanding that the $Z_2 \otimes Z_2'$ symmetry is spontaneously broken by vev of scalar fields that preferentially couple to well defined flavors, L_e and $L_\mu \pm L_\tau$.

If one imposes only the first Z_2 and disregards Z_2', extra off-diagonal entries $m_{13} = m_{23}/\varphi$ are allowed in the neutrino mass matrix. This does not affect θ_{12} and produces a non-zero θ_{13} correlated with a deviation of θ_{23} from maximal mixing: neglecting the solar masses, and at first order in $\theta_{13} \ll 1$, the relation is

$$\theta_{13} \simeq (\theta_{23} - \pi/4)/\varphi. \tag{5}$$

Better models can maybe be built using the icosahedral group, isomorphic to A_5, the group of even permutations of 5 objects. Indeed this group is connected to φ, as can be seen either geometrically, or by noticing that the permutation factor $\omega = e^{2\pi i/5}$ satisfies $\varphi = 1 + \omega + \omega^4$. The A_4 group has been used to justify bi-tri-maximal mixing, because it allows to justify factors of $\omega = e^{2\pi i/3}$ in the mass matrices [3].

Noticing that the prediction (2) satisfies with high accuracy the 'quark/lepton complementarity' [8, 9], i.e. the observation that $\theta_{12} + \theta_C$ is numerically close to $\pi/4$, where θ_C is the Cabibbo angle, motivates us to give a golden geometric explanation also to the Cabibbo angle. As previously discussed, $\sin^2 2\theta_{12} = 4/5$ arises if m_ν is symmetric with respect to the reflection around the diagonal of golden rectangle in the (12) plane, and λ_E is diagonal in the first two generations. SU(5) unification relates

the down-quark Yukawa matrix λ_D to λ_E and suggests that the up-quark Yukawa matrix λ_U is symmetric, like m_ν. We therefore assume that λ_D is diagonal in the two first generations and that λ_U is invariant under a Z_2 reflection described by a matrix analogous to R in eq. (4), but with the factors $1 \leftrightarrow 2$ exchanged. Fig. 1 illustrates the geometrical meaning of two reflection axis (dashed lines): the up-quark reflection is along the diagonal of the golden rectangle tilted by $\pi/4$; notice also the connection with the decomposition of the golden rectangle as an infinite sum of squares ('golden spiral'). Similarly to the neutrino case, this symmetry allows for two independent terms that can be tuned such that $m_u \ll m_c$:

$$\lambda_U = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\lambda}{\sqrt{5}} \begin{pmatrix} -2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{6}$$

The second term fixes $\cot\theta_C = \varphi^3$, as can be geometrically seen from fig. 1. We therefore have

$$\sin^2 2\theta_C = 1/5 \qquad \text{i.e.} \qquad \theta_{12} + \theta_C = \pi/4 \qquad \text{i.e.} \qquad V_{us} = \sin\theta_C = (1+\varphi^6)^{-1/2} = 0.229. \tag{7}$$

This prediction is 1.9σ above the present best-fit value, $\sin\theta_C = 0.2258 \pm 0.0021$ [10]. We expect that the golden prediction for V_{us} has an uncertainty at least comparable to $|V_{ub}| \sim |V_{td}| \sim \text{few} \cdot 10^{-3}$.

The textures we proposed are not stable under quantum corrections (this is why flavor symmetries behind them must be broken), and presumably hold at some high scale of flavour generation, possibly of order $M_{\rm GUT}$ or $M_{\rm Pl}$. Within the Standard Model, RGE corrections to θ_{12} and θ_{23} are numerically small, at the per-mille level, because suppressed by λ_τ^2. The RGE corrections from new physics are model dependent and could be larger in models with larger flavor dependent Yukawas [11], such as supersymmetric seesaw model at large $\tan\beta$.

In conclusion, we have presented a simple texture (as well as its seesaw model realization) that might be behind the non-trivial value of the solar mixing angle. The prediction $\theta_{12} = 31.7^\circ$ is 1.4σ below the present best-fit value and 2.6σ away from pure tri-bi-maximal mixing. These two possibilities might be discriminated in the near future, as the SNO (phase 3) and KamLAND experiments already collected new data that should reduce the uncertainty on θ_{12} [12][1]. We identified a $Z_2 \otimes Z_2'$ symmetry behind the neutrino mass matrix, related to a reflection of the first two generations along the diagonal of the golden rectangle. This symmetry allows for an extra flavour-diagonal contribution to the mass matrices, and consequently for $0\nu 2\beta$ decay. Assuming that a similar golden rectangle structure also controls flavor mixing among the first two generation of quarks, we get a geometric prediction for the Cabibbo angle, $\theta_C = \pi/4 - \theta_{12} = 13.3^\circ$. Although the golden predictions suffer from the uncertainties related to $(1,3)$ mixings in both sectors, we quoted those values with 3 digits because they might optimistically hold up to few per mille precision.

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[1] Including new KamLAND data presented by I. Shimizu at the 10th Conference on Topics in Astroparticle and Underground Physics, the global fit of KamLAND and SNO data now is $\tan^2\theta_{12} = 0.49 \pm 0.06$ assuming $\theta_{13} = 0$.

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